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2009 J. Phys. A: Math. Theor. 42 205201

(http://iopscience.iop.org/1751-8121/42/20/205201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 205201 (8pp) doi:10.1088/1751-8113/42/20/205201

Polynomial solutions of nonlinear integral equations

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Received 1 March 2009, in final form 2 April 2009
Published 28 April 2009
Online at stacks.iop.org/JPhysA/42/205201

Abstract
We analyze the polynomial solutions of a nonlinear integral equation, generalizing the work of Bender and Ben-Naim (2007 *J. Phys. A: Math. Theor.* **40** F9, 2008 *J. Nonlinear Math. Phys.* **15** (Suppl. 3) 73). We show that, in some cases, an orthogonal solution exists and we give its general form in terms of kernel polynomials.

PACS numbers: 02.30.Gp, 02.30.Rz, 02.30.Mv
Mathematics Subject Classification: 45G10, 33C45

1. Introduction

In [2], Bender and Ben-Naim studied the polynomial solutions of the nonlinear integral equation

$$\int_a^b P(y)P\,(x+y)\,\omega(y)\,\mathrm{d}y = P(x). \tag{1}$$

They showed that the solutions $P_n(x)$ are orthogonal with respect to the measure $x\omega(x)$ and considered other equations of the form

$$\int_a^b P(y)P\,[F(x,y)]\,\omega(y)\,\mathrm{d}y = P(x), \tag{2}$$

with

$$F(x,y) = xy, \qquad x + a_1 + a_2 y \quad \text{and} \quad x + f(y)$$

(see http://staff.science.uva.nl/∼thk/art/comment/BenderComment.pdf for Tom Koornwinder's comment on their paper). They continued their investigation in [3], where they used limit relations and asymptotic properties of the Laguerre and Jacobi polynomials to obtain some interesting integral identities.

The purpose of this paper is to generalize their results to the case $F(x,y) = \alpha(y) + x\beta(y)$ for arbitrary functions $\alpha(y)$ and $\beta(y)$ and to understand the nature of the families of orthogonal polynomials that arise as solutions of (1).

2. General case

Let $\omega(y)$ be a non-negative integrable function on the interval (a, b), such that

$$\int_a^b \omega(y)\,\mathrm{d}y = 1 \tag{3}$$

and let $\mathcal{L}_\omega$ be the linear functional defined by

$$\mathcal{L}_\omega[f] = \int_a^b f(y)\omega(y)\,\mathrm{d}y. \tag{4}$$

We say that a sequence of polynomials (P_n) is an orthogonal polynomial sequence (OPS) with respect to $\mathcal{L}_\omega$ if [4]

(1) $P_n(x)$ is a polynomial of degree n.
(2) $\mathcal{L}_\omega[P_n P_m] = h_n\delta_{n,m},\ n, m = 0, 1, \ldots,$
where $h_n \neq 0$ for all n and $\delta_{n,m}$ is Kronecker's delta.

To warranty the existence of a polynomial sequence solution (P_n), we consider the special form of equation (2)

$$\int_a^b P_n(y)P_n[\alpha(y) + x\beta(y)]\,\omega(y)\,\mathrm{d}y = P_n(x), \tag{5}$$

where $\alpha(y)$ and $\beta(y)$ are integrable functions on (a, b).

Example 1. Let

$$\omega(y) = \tfrac{3}{2}y^2, \qquad a = -1, b = 1, \qquad \alpha(y) = \tfrac{5}{3}y, \qquad \beta(y) = \mu \neq 0.$$

Then, we have

$$P_0(x) = 1, \qquad P_1(x) = \frac{1}{\mu} \pm \frac{\sqrt{\mu - 1}}{\mu}x, \ldots.$$

Example 2. Let

$$\omega(y) = \tfrac{3}{2}y^2, \qquad a = -1, b = 1, \qquad \alpha(y) = \tfrac{3}{20}\mu y, \qquad \beta(y) = y.$$

Then, we have

$$P_0(x) = 1, \qquad P_1(x) = \frac{1 \pm \sqrt{1-\mu}}{2} + \frac{5}{3}x, \ldots.$$

The previous examples illustrate how, even for simple functions, the integral equation (5) can have unique or multiple solutions which are real or complex depending on the choice of the parameter μ.

Writing

$$P_n(x) = \sum_{k=0}^{n} c_k x^k, \tag{6}$$

we have

$$P_n[\alpha(y) + x\beta(y)] = \sum_{k=0}^{n} \gamma_k(y)\,x^k, \tag{7}$$

where

$$\gamma_k(y) = \beta^k(y)\sum_{j=k}^{n} c_j \binom{j}{k} \alpha^{j-k}(y).$$

Using (6) and (7) in (5), we get

$$\mathcal{L}_\omega\left[P_n\gamma_k\right] = c_k, \qquad 0 \leqslant k \leqslant n. \tag{8}$$

Introducing the matrix **A** defined by

$$\mathbf{A} = \begin{bmatrix} \binom{0}{0}\mathcal{L}_\omega[P_n] & \binom{1}{0}\mathcal{L}_\omega[P_n\alpha] & \binom{2}{0}\mathcal{L}_\omega[P_n\alpha^2] & \cdots & \binom{n}{0}\mathcal{L}_\omega[P_n\alpha^n] \\ 0 & \binom{1}{1}\mathcal{L}_\omega[P_n\beta] & \binom{2}{1}\mathcal{L}_\omega[P_n\alpha\beta] & \cdots & \binom{n}{1}\mathcal{L}_\omega[P_n\alpha^{n-1}\beta] \\ 0 & 0 & \binom{2}{2}\mathcal{L}_\omega[P_n\beta^2] & \cdots & \binom{n}{2}\mathcal{L}_\omega[P_n\alpha^{n-2}\beta^2] \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n}\mathcal{L}_\omega[P_n\beta^n] \end{bmatrix} \tag{9}$$

and the vector

$$\mathbf{C}^T = [c_0, \ \cdots \ c_n],$$

we see from (8) that **C** is an eigenvector of **A** with corresponding eigenvalue 1.

Therefore, to have a solution **C** different from the zero vector, it must be true that $\mathcal{L}_\omega[P_n\beta^k] = 1$ for some $0 \leqslant k \leqslant n$. Note that if we impose the condition $\mathcal{L}_\omega[P_n\beta^n] = 1$, then the vector $\mathbf{C}^T$=[0, $\cdots$ 0, c_n] is always an eigenvector of **A**. However, this leads to trivial sequences of the form $P_n(x) = c_n x^n$.

A possible non-trivial solution of the equation $\mathbf{AC} = \mathbf{C}$ is to take $\mathbf{A} = \mathbf{I}$, where **I** denotes the identity matrix. Thus, we require that

$$\mathcal{L}_\omega[P_n\alpha^{j-i}\beta^i] = \delta_{i,j}, \qquad 0 \leqslant i \leqslant j \leqslant n. \tag{10}$$

Since (10) is a system of $\binom{n+2}{2}$ equations with $n + 1$ unknowns, it admits (if any) infinitely many solutions. In order to have a unique solution, we consider the following cases:

(1) $\alpha(y) = 0, \qquad \beta(y) \neq 1.$

We see from (9) that for **A** to be equal to the identity matrix, we need to have

$$\mathcal{L}_\omega[P_n\beta^i] = 1, \qquad 0 \leqslant i \leqslant n, \tag{11}$$

which, using (3), we can rewrite as

$$\mathcal{L}_\omega[P_n(1 - \beta^i)] = 0, \qquad 0 \leqslant i \leqslant n,$$

or

$$\mathcal{L}_{(\beta-1)\omega}\left[P_n\frac{(\beta^i - 1)}{\beta - 1}\right] = 0, \qquad 0 \leqslant i \leqslant n.$$

Thus, (11) is equivalent to

$$\mathcal{L}_\omega[P_n] = 1, \qquad \mathcal{L}_{(\beta-1)\omega}[P_n\beta^i] = 0, \qquad 0 \leqslant i \leqslant n - 1. \tag{12}$$

If $\beta(y)$ is linear, it follows from (12) that (P_n) will be a sequence of orthogonal polynomials with respect to the linear functional $\mathcal{L}_{(\beta-1)\omega}$, provided that $\mathcal{L}_{(\beta-1)\omega}\left[P_n\beta^n\right] \neq 0$. For this last condition to be true, $\beta\left(y\right) - 1$ must not vanish in the interval (a, b). Hence, $\beta\left(y\right)$ should be of the form

$$\beta\left(y\right) = \sigma\left(y - \zeta\right) + 1,$$

with $\sigma \neq 0$ and $\zeta \notin (a, b)$.

(2) $\alpha(y) \neq 0, \qquad \beta(y) = 1.$
In this case, we must impose that

$$\mathcal{L}_{\omega}[P_n] = 1, \qquad \mathcal{L}_{\omega}[P_n \alpha^i] = 0, \qquad 1 \leqslant i \leqslant n,$$

or, equivalently,

$$\mathcal{L}_{\omega}[P_n] = 1, \qquad \mathcal{L}_{\alpha\omega}[P_n \alpha^i] = 0, \qquad 0 \leqslant i \leqslant n-1. \tag{13}$$

If

$$\alpha(y) = \tau(y - \varsigma),$$

with $\tau \neq 0$ and $\varsigma \notin (a, b)$, the polynomials $P_n(x)$ will be orthogonal with respect to $\mathcal{L}_{\alpha\omega}$.

We summarize the results of this section in the following theorems.

Theorem 3. *Let $\beta(y) \neq 1$ on (a, b) and suppose that $\Delta_n \neq 0$ for all n, with*

$$\Delta_n = \begin{vmatrix} 1 & \mathcal{L}_{\omega}[y] & \cdots & \mathcal{L}_{\omega}[y^n] \\ \mathcal{L}_{(\beta-1)\omega}[1] & \mathcal{L}_{(\beta-1)\omega}[y] & \cdots & \mathcal{L}_{(\beta-1)\omega}[y^n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{(\beta-1)\omega}[\beta^{n-1}] & \mathcal{L}_{(\beta-1)\omega}[y\beta^{n-1}] & \cdots & \mathcal{L}_{(\beta-1)\omega}[y^n\beta^{n-1}] \end{vmatrix}. \tag{14}$$

If (P_n) is defined by

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} 1 & x & \cdots & x^n \\ \mathcal{L}_{(\beta-1)\omega}[1] & \mathcal{L}_{(\beta-1)\omega}[y] & \cdots & \mathcal{L}_{(\beta-1)\omega}[y^n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{(\beta-1)\omega}[\beta^{n-1}] & \mathcal{L}_{(\beta-1)\omega}[y\beta^{n-1}] & \cdots & \mathcal{L}_{(\beta-1)\omega}[y^n\beta^{n-1}] \end{vmatrix}, \tag{15}$$

then,

$$\int_a^b P_n(y) P_n\left[\beta(y)x\right] \omega(y)\, \mathrm{d}y = P_n(x)$$

for all n.

Proof. It is clear from (14) and (15) that

$$\mathcal{L}_{\omega}[P_n] = 1, \qquad \mathcal{L}_{(\beta-1)\omega}[P_n \beta^i] = 0, \qquad 0 \leqslant i \leqslant n-1$$

for all $n \geqslant 1$. We have

$$\begin{aligned} &\int_a^b P_n(y) P_n\left[\beta(y)x\right] \omega(y)\, \mathrm{d}y - P_n(x) \\ &\qquad = \int_a^b P_n(y) P_n\left[\beta(y)x\right] \omega(y)\, \mathrm{d}y - \int_a^b P_n(y) P_n(x) \omega(y)\, \mathrm{d}y \\ &\qquad = \int_a^b P_n(y) \left\{P_n\left[\beta(y)x\right] - P_n(x)\right\} \omega(y)\, \mathrm{d}y. \end{aligned}$$

Using (6), we get

$$\begin{aligned} &\int_a^b P_n(y) P_n[\beta(y)x] \omega(y)\, \mathrm{d}y - P_n(x) = \sum_{k=1}^{n} c_k \mathcal{L}_{\omega}[P_n(\beta^k - 1)] x^k \\ &\qquad = \sum_{k=1}^{n} c_k \mathcal{L}_{(\beta-1)\omega}\left[P_n \frac{(\beta^k - 1)}{\beta - 1}\right] x^k = \sum_{k=1}^{n} c_k \left[\sum_{j=0}^{k-1} \mathcal{L}_{(\beta-1)\omega}[P_n \beta^j]\right] x^k = 0. \end{aligned}$$

□

Theorem 2. *Let* $\alpha(y) \neq 0$ *on* (a, b) *and suppose that* $\Delta_n \neq 0$ *for all* n*, with*

$$\Delta_n = \begin{vmatrix} 1 & \mathcal{L}_\omega[y] & \cdots & \mathcal{L}_\omega[y^n] \\ \mathcal{L}_{\alpha\omega}[1] & \mathcal{L}_{\alpha\omega}[y] & \cdots & \mathcal{L}_{\alpha\omega}[y^n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{\alpha\omega}[\alpha^{n-1}] & \mathcal{L}_{\alpha\omega}[y\alpha^{n-1}] & \cdots & \mathcal{L}_{\alpha\omega}[y^n\alpha^{n-1}] \end{vmatrix}. \tag{16}$$

If (P_n) *is defined by*

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} 1 & x & \cdots & x^n \\ \mathcal{L}_{\alpha\omega}[1] & \mathcal{L}_{\alpha\omega}[y] & \cdots & \mathcal{L}_{\alpha\omega}[y^n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{\alpha\omega}[\alpha^{n-1}] & \mathcal{L}_{\alpha\omega}[y\alpha^{n-1}] & \cdots & \mathcal{L}_{\alpha\omega}[y^n\alpha^{n-1}] \end{vmatrix}, \tag{17}$$

then,

$$\int_a^b P_n(y) P_n[\alpha(y) + x]\, \omega(y)\, dy = P_n(x)$$

for all n.

Proof. It is clear from (14) and (15) that

$$\mathcal{L}_\omega[P_n] = 1, \qquad \mathcal{L}_{\alpha\omega}[P_n\alpha^i] = 0, \qquad 0 \leqslant i \leqslant n-1$$

for all $n \geqslant 1$. We have

$$\int_a^b P_n(y) P_n[\alpha(y) + x]\, \omega(y)\, dy - P_n(x) = \int_a^b P_n(y) \{P_n[\alpha(y) + x] - P_n(x)\}\, \omega(y)\, dy$$
$$= \sum_{k=1}^{n} q_k(x) \mathcal{L}_\omega[P_n\alpha^k] = \sum_{k=0}^{n-1} q_{k+1}(x) \mathcal{L}_{\alpha\omega}[P_n\alpha^k] = 0,$$

where we have used (6) and the polynomials $q_k(x)$ are defined by

$$q_k(x) = \sum_{j=k}^{n} c_j \binom{j}{k} x^{j-k}.$$

□

Theorem 3. *Let* $\zeta \notin (a, b)$ *and* (P_n) *be an OPS for* $\mathcal{L}_{(y-\zeta)\omega}$ *satisfying*

$$\mathcal{L}_\omega[P_n] = 1, \qquad n = 0, 1, \ldots. \tag{18}$$

Then,

$$\int_a^b P_n(y) P_n[(y - \zeta)(\tau + \sigma x) + x]\, \omega(y)\, dy = P_n(x). \tag{19}$$

Proof. Using (18), we see that

$$\int_a^b P_n(y) P_n[(y - \zeta)(\tau + \sigma x) + x] \omega(y)\, dy - P_n(x)$$
$$= \int_a^b P_n(y) \{P_n[(y - \zeta)(\tau + \sigma x) + x] - P_n(x)\} \omega(y)\, dy$$
$$= \sum_{k=1}^{n} q_k(x) (\tau + \sigma x)^k \mathcal{L}_\omega[(y - \zeta)^k P_n]$$
$$= \sum_{k=0}^{n-1} q_{k+1}(x) (\tau + \sigma x)^{k+1} \mathcal{L}_{(y-\zeta)\omega}[(y - \zeta)^k P_n] = 0.$$

□

The two main cases considered in [2], correspond to the special values

$$\zeta = 0, \tau = 1, \sigma = 0, \qquad \text{for} \quad P_n(y)P_n(x+y)$$
$$\zeta = 1, \tau = 0, \sigma = 1, \qquad \text{for} \quad P_n(y)P_n(xy).$$

Unfortunately, the reciprocal of theorem 3 is not true in general. For example, taking

$$\omega(y) = \mathrm{e}^{-y}, \qquad a = 0, \qquad b = \infty, \qquad \zeta = 0, \qquad \tau = 1, \qquad \sigma = 1,$$

we get as a possible solution of (19)

$$P_0(x) = 1, \qquad P_1(x) = 2 - x, \qquad P_2(x) = \tfrac{7}{5} - \tfrac{1}{5}x - \tfrac{1}{10}x^2,$$
$$P_3(x) = \tfrac{43}{17} - \tfrac{32}{17}x + \tfrac{3}{34}x^2 + \tfrac{1}{34}x^3, \ldots.$$

We have

$$\mathcal{L}_{y\omega}[P_1] = 0, \qquad \mathcal{L}_{y\omega}[P_2] = \tfrac{2}{5}, \qquad \mathcal{L}_{y\omega}[P_3] = 0, \qquad \mathcal{L}_{y\omega}[yP_3] = -\tfrac{10}{17} \cdots$$

and therefore (P_n) is not an OPS for $\mathcal{L}_{(y-\zeta)\omega}$.

In the following section, we shall see that the only solutions of (19) which are an OPS for $\mathcal{L}_{(y-\zeta)\omega}$, consist of the so-called kernel polynomials corresponding to $\mathcal{L}_\omega$.

3. Kernel polynomials

Let $(\mathfrak{p}_n)$ be the sequence of orthonormal polynomials with respect to $\mathcal{L}_\omega$ defined by (4). The kernel polynomials $K_n(x;\zeta)$ corresponding to $\mathcal{L}_\omega$ with parameter ζ are defined by [6]

$$K_n(x;\zeta) = \sum_{k=0}^{n} \frac{\mathfrak{p}_k(\zeta)}{\mathcal{L}_\omega\left[\mathfrak{p}_k^2\right]}\mathfrak{p}_k(x), \tag{20}$$

where $\mathfrak{p}_n(\zeta) \neq 0$ for all n. Using the Christoffel–Darboux identity [1], we have

$$K_n(x;\zeta) = \frac{1}{\mathcal{L}_\omega\left[\mathfrak{p}_n^2\right]} \frac{\mathfrak{p}_{n+1}(x)\,\mathfrak{p}_n(\zeta) - \mathfrak{p}_n(x)\,\mathfrak{p}_{n+1}(\zeta)}{x-\zeta}.$$

The kernel polynomials $K_n(x;\zeta)$ have the following properties [4]:

(1) They are orthogonal with respect to the functional $\mathcal{L}_{(y-\zeta)\omega}$.
(2) They have the reproducing property

$$\mathcal{L}_\omega[K_n(y;\zeta)p_n(y)] = p_n(\zeta),$$

for any polynomial $p_n(x)$ of degree less or equal than n.

It follows that, up to a multiplicative constant λ, the kernel polynomials $K_n(x;\zeta)$ are solutions of (1). To find λ we use (18) and obtain

$$1 = \mathcal{L}_\omega[\lambda K_n(x;\zeta)] = \lambda \sum_{k=0}^{n} \frac{\mathfrak{p}_k(\zeta)}{\mathcal{L}_\omega\left[\mathfrak{p}_k^2\right]}\mathcal{L}_\omega[\mathfrak{p}_k(x)] = \lambda.$$

Thus, we have the following result.

Corollary 6. *The only OPS (P_n) which is a solution of the nonlinear integral equation*

$$\int_a^b P_n(y)P_n[(y-\zeta)(\tau+\sigma x)+x]\,\omega(y)\,\mathrm{d}y = P_n(x)$$

is $P_n(x) = K_n(x;\zeta)$, where $K_n(x;\zeta)$ is defined by (20).

Example 7. Let

$$\omega(y) = \tfrac{1}{2}, \qquad a = -1, \qquad b = 1, \qquad \zeta = 1, \qquad \tau = 1, \qquad \sigma = 0.$$

Then, we have

$$P_0(x) = 1, \qquad P_1(x) = 1 + 3x, \qquad P_2(x) = -\tfrac{3}{2} + 3x + \tfrac{15}{2}x^2, \ldots.$$

If we denote by $\mathbf{P}_n(x)$ the Legendre polynomials, defined by [5]

$$\mathbf{P}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2}\right)^k,$$

then it follows from (20) that

$$P_n(x) = \sum_{k=0}^{n} \frac{\mathbf{P}_k(1)}{(2k+1)^{-1}} \mathbf{P}_k(x) = \sum_{k=0}^{n} (2k+1)\, \mathbf{P}_k(x).$$

Example 8. Again, let

$$\omega(y) = \mathrm{e}^{-y}, \qquad a = 0, \qquad b = \infty, \qquad \zeta = 0, \qquad \tau = 1, \qquad \sigma = 1.$$

Then,

$$P_n(x) = \sum_{k=0}^{n} L_k(0) L_k(x) = \sum_{k=0}^{n} L_k(x),$$

(P_n) is an OPS for $\mathcal{L}_{(y-\zeta)\omega}$, where

$$L_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} (-x)^k$$

denotes the Laguerre polynomial [5]. We have

$$P_0(x) = 1, \qquad P_1(x) = 2 - x, \qquad P_2(x) = 3 - 3x + \tfrac{1}{2}x^2,$$
$$P_3(x) = 4 - 6x + 2x^2 - \tfrac{1}{6}x^3, \ldots.$$

4. Concluding remarks

We have studied the polynomial solutions of the nonlinear integral equation (5). We have shown that, in some cases, a solution which is an OPS exists and we have given the general form of these orthogonal solutions.

However, much remains to be discovered about the solutions of (5). A few questions that come to mind are:

(1) For which choice of α and β will there be a unique solution?
(2) Is it possible to describe all possible solutions?
(3) For which values of ζ, τ and σ will the solution of (19) be unique? It seems that for this to be true, one needs to consider the symmetric case, when

$$\zeta\sigma + \tau = 1.$$

Is this condition sufficient?

We hope that other researchers will find this problem interesting and continue its analysis.

Acknowledgments

We want to thank Ted Chihara, Mourad Ismail, Tom Koornwinder and Robert Maier for useful comments on earlier drafts of this paper. This work was completed while D Dominici was visiting Technische Universität Berlin and supported in part by a Sofja Kovalevskaja Award from the Humboldt Foundation, provided by Professor Olga Holtz. He wishes to thank Olga for her generous sponsorship and his colleagues at TU Berlin for their continuous help.

References

[1] Andrews G E, Askey R and Roy R 1999 Special functions *Encyclopedia of Mathematics and its Applications* vol 71 (Cambridge: Cambridge University Press)

[2] Bender C M and Ben-Naim E 2007 Nonlinear integral-equation formulation of orthogonal polynomials *J. Phys. A: Math. Theor.* **40** F9–15

[3] Bender C M and Ben-Naim E 2008 Nonlinear-integral-equation construction of orthogonal polynomials *J. Nonlinear Math. Phys.* **15** (Suppl. 3) 73–80

[4] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach Science Publishers)

[5] Ismail M E H 2005 Classical and quantum orthogonal polynomials in one variable *Encyclopedia of Mathematics and its Applications* vol 98 (Cambridge: Cambridge University Press)

[6] Szegö G 1975 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)